

Quantum Mechanical Operator of Time

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Abstract

The self adjoint operator of time in non-relativistic quantum mechanics is found within the approach where the ordinary Hamiltonian is not taken to be conjugate to time. The operator version of the re-expressed Liouville equation with the total Hamiltonian, consisting of the part that is a conventional function of coordinate and momentum and the part that is conjugate to time, is considered. The von Neumann equation with quantized time is found and discussed from the point of view of exact time measurement.

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1 Introduction

Time in quantum mechanics (QM), as is well known, appears as a c-number parameter, not as an operator representing observable. In this way QM differs from special relativity where time and space coordinates are treated on an equal footing. Absence of time operator in QM is strongly related to the finite lower bound of the energy spectrum. Namely, according to Pauli, it is not possible to find self adjoint operator which is canonically conjugate to Hamiltonian for it has bounded-from-below spectra. Since this objection, there have been many attempts to address time and/or its operator in QM. The extensive lists of references one can find in [1-2], while the present approach have some concrete similarities with those given in [3-7].

Here, operator of time \hat{t} is found together with its conjugate one \hat{s} and these two emerged after their classical mechanics (CM) counterparts (t and

s) have been analyzed. As would become obvious, appearance of \hat{t} and \hat{s} is consistent with the standard QM and \hat{t} is as was expected - self adjoint and with continuous spectra.

This article is organized as follows. In Sec. II. discussion of the position of time in CM is given. In Sec. III. the self adjoint time operator, new form of dynamical equation of QM and possibility of exact measurement of time in QM are analyzed in details. Finally, in Sec. IV. some remarks are given.

2 Time and Dynamical Equation of Classical Mechanics

If time and space coordinates should be on an equal footing, then there should be operator of time \hat{t} which has similar commutation relation with its conjugate operator, say \hat{s} , as coordinate \hat{q} has with \hat{p} . (*Nota bene*, it is not guaranteed *a priori* that the Hamiltonian is conjugate to time, so that is the reason for taking \hat{s} .) If the Lie bracket of QM, which is $\frac{1}{i\hbar}[\ , \]$, of \hat{t} and \hat{s} does not vanish, then the Poisson bracket, being the Lie bracket of CM, should not vanish for t and s , the last being the CM counterpart of \hat{s} . This is necessary since there should be 1-1 correspondence between QM and CM. On the other hand, perhaps the most important place in CM where Poisson bracket appears is the Liouville equation:

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} = \frac{\partial H}{\partial q} \cdot \frac{\partial \rho}{\partial p} - \frac{\partial H}{\partial p} \cdot \frac{\partial \rho}{\partial q}. \quad (1)$$

In this expression derivation with respect to s is not (manifestly) present, so (it seems that) time and coordinate are not on an equal footing since there are $\frac{\partial}{\partial p}$ beside $\frac{\partial}{\partial q}$. However, if $\frac{\partial H}{\partial s} = 1$, $\frac{\partial H}{\partial t} = 0$ and $\rho = \rho(q, p, t, s)$, then the LHS of (1) is equal to:

$$\frac{\partial H}{\partial s} \cdot \frac{\partial \rho}{\partial t} - \frac{\partial H}{\partial t} \cdot \frac{\partial \rho}{\partial s}. \quad (2)$$

If (2) is taken instead of the LHS of (1), then the complete Liouville equation is:

$$\begin{aligned} \frac{\partial H}{\partial s} \cdot \frac{\partial \rho}{\partial t} - \frac{\partial H}{\partial t} \cdot \frac{\partial \rho}{\partial s} = \\ = \frac{\partial H}{\partial q} \cdot \frac{\partial \rho}{\partial p} - \frac{\partial H}{\partial p} \cdot \frac{\partial \rho}{\partial q}. \end{aligned} \quad (3)$$

Now, one can introduce:

$$\{, \}_{q,p} = \frac{\partial}{\partial q} \cdot \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \cdot \frac{\partial}{\partial q}, \quad (4)$$

$$\{, \}_{t,s} = -\frac{\partial}{\partial t} \cdot \frac{\partial}{\partial s} + \frac{\partial}{\partial s} \cdot \frac{\partial}{\partial t}, \quad (5)$$

and

$$\{, \}_W = \{, \}_{q,p} - \{, \}_{t,s}, \quad (6)$$

which act on ordered pair $(A(q, p, t, s), B(q, p, t, s))$. Obviously, in (6) coordinate and time are on an equal footing and one can reexpress (3) as:

$$\{H, \rho\}_W = 0. \quad (7)$$

So, if $\frac{\partial H}{\partial s} = 1$ coordinate and time can be equally treated in the generalized Liouville equation (7). Therefore, it is necessary to deduce the relation between H and s . The first choice is $H = s$, the meaning of which is that Hamilton function is the conjugate variable to time in CM. But, this would imply that the Hamiltonian is the conjugate observable to time in QM. Assuming this, one would find oneself faced with the problem of time in QM, which was mentioned above. Therefore, H is not equal to s , but:

$$H = H(q, p) + s. \quad (8)$$

With this form of Hamilton function one should proceed in quantization in order to avoid collision with well known facts of QM. But, before addressing quantization of the above proposed formalism, few comments are in order. Due to (8), s is present in (7), but it is ineffective in sense that solutions of this dynamical equation are:

$$\rho(q, p, t, s) = \rho(q, p, t) \delta(s - s_o). \quad (9)$$

$\delta(s - s_o)$ in (9) ensures that $\rho(q, p, t, s)$ is pure state in case when $\rho(q, p, t) = \delta(q - q(t)) \delta(p - p(t))$. If for $t = t_a$ one calculates the mean value of $H(q, p)$ according to:

$$\int \int \int H(q, p) \rho(q, p, t_a, s) dq dp ds, \quad (10)$$

then it would be the same as the mean value of $H(q, p)$ calculated for $\rho(q, p, t_a)$ in standard phase space formalism of CM, where s is not considered. On the other hand, if one calculates the mean value of $H(q, p) + s$

for (9), then it would differ from (10) in additional s_o . However, the measurement of energy is formalized by the application of $H(q, p)$, not but by $H(q, p) + s$, so one can take $s_o = 0$ since this value is irrelevant for physics.

3 Time and Dynamical Equation of Quantum Mechanics

Regarding QM, there should be \hat{t} which has continuous spectra. Beside this operator, there should be \hat{s} (and \hat{q} and \hat{p}). Since:

$$\{t, s\}_{t,s} = -1,$$

it should be:

$$\frac{1}{i\hbar}[\hat{t}, \hat{s}] = -1. \quad (11)$$

On the other side, existence of $q, p, t, s, \{, \}_{q,p}$ and $\{, \}_{t,s}$ resembles the situation within standard CM when the system with two degrees of freedom is under consideration (when there are q_x, p_x, q_y and p_y). So, one can quantize the above given formalism following Dirac and his procedure appropriate for the case of two degrees of freedom. This means that one should take $\mathcal{H}_{space} \otimes \mathcal{H}_{time}$ for the space where operators $\hat{q} \otimes \hat{I}, \hat{p} \otimes \hat{I}, \hat{I} \otimes \hat{t}$ and $\hat{I} \otimes \hat{s}$ act. Due to (11), operators \hat{t} and \hat{s} in $|t\rangle$ and $|s\rangle$ representations should be t and $i\hbar \frac{\partial}{\partial t}$ and $-i\hbar \frac{\partial}{\partial s}$ and s , respectively. Instead of CM Hamilton function $H = H(q, p) + s$ there should be its QM counterpart:

$$\hat{H} = H(\hat{q} \otimes \hat{I}, \hat{p} \otimes \hat{I}) + \hat{I} \otimes \hat{s} = H(\hat{q}, \hat{p}) \otimes \hat{I} + \hat{I} \otimes \hat{s}. \quad (12)$$

Except in the case of free particle, the first part of this Hamiltonian has, as in standard QM, discrete bounded-from-below spectrum E_i , while the second part has the continuous s .

What remains in transition from CM to QM is to declare what is dynamical equation within this approach. Instead of imitating procedure related to von Neumann equation in case of two degrees of freedom, dynamical equation of QM will be reached in another way.

Without going into details, in [8] it was shown that it is possible to introduce symmetrized product (ordering rule), denoted by \circ , in standard

QM in such a way that:

$$\frac{1}{i\hbar}[\hat{H}, \hat{\rho}] = \frac{\partial \hat{H}}{\partial \hat{q}} \circ \frac{\partial \hat{\rho}}{\partial \hat{p}} - \frac{\partial \hat{H}}{\partial \hat{p}} \circ \frac{\partial \hat{\rho}}{\partial \hat{q}}, \quad (13)$$

where $\hat{H} = \sum_{m,n} \hat{q}^m \circ \hat{p}^n$. Immediate consequence of (13) is that the von Neumann equation becomes the operator version of the Liouville equation. Hence, the generalized Liouville equation (7) is the classical mechanical counterpart of the operator version of generalized Liouville equation:

$$\{\hat{H}, \hat{\rho}\}_W = 0, \quad (14)$$

within which derivations are with respect to the above given operators, product is the symmetrized one, defined in [8], and where \hat{H} is given by (12) while $\hat{\rho}$ is statistical operator in $\mathcal{H}_{space} \otimes \mathcal{H}_{time}$. As was the case for (7), in (14) time and coordinate are treated equally. From (14) one gets:

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} = \frac{1}{i\hbar}[H(\hat{q}, \hat{p}), \hat{\rho}], \quad (15)$$

where $H(\hat{q}, \hat{p})$ is the first term on the RHS of (12). The last equation is the von Neumann equation with quantized time.

Due to (11), the equation (15) can be reexpressed:

$$[\hat{s}, \hat{\rho}] = [H(\hat{q}, \hat{p}), \hat{\rho}]. \quad (16)$$

The solutions of (16) are:

$$\begin{aligned} \hat{\rho} &= \sum_{i,j} c_{ij} |E_i\rangle \langle E_j| \otimes e^{\frac{1}{i\hbar} s_i \hat{t}} |s_o\rangle \langle s_o| e^{-\frac{1}{i\hbar} s_j \hat{t}} = \\ &= \sum_{i,j} c_{ij} |E_i\rangle \langle E_j| \otimes |s'_i\rangle \langle s'_j|, \end{aligned} \quad (17)$$

where $H(\hat{q}, \hat{p})|E_i\rangle = E_i|E_i\rangle$ and:

$$s_i - s_j = E_i - E_j. \quad (18)$$

In case when the system under consideration is in a pure state, $\hat{\rho}$ becomes:

$$|\psi\rangle \langle \psi| = \sum_i c_i |E_i\rangle \otimes |s_i\rangle \sum_j c_j^* \langle E_j| \otimes \langle s_j|. \quad (19)$$

After substituting this state in (16) and after taking coordinate-time $|q\rangle \otimes |t\rangle$ representation, one arrives to:

$$i\hbar \frac{\partial}{\partial t} \psi(q, t) = H(q, -i\hbar \frac{\partial}{\partial q}) \psi(q, t), \quad (20)$$

where $\psi(q, t) = (\langle q| \otimes \langle t|) |\psi\rangle$. Obviously, (20) is the Schroedinger equation.

Without affecting physical meaning, one can simplify expression (17) by taking $s_o = 0$ and $s_i = E_i$. Then, in a case when system is in state with sharp value of energy, $\hat{\rho}$ becomes $|\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle = |E_i\rangle \otimes |E_i\rangle$. Needless to say, these two $|E_i\rangle$'s are pretty much different. The first one is the element of the Hilbert space \mathcal{H}_{space} and it is normalized to one, while the second is the element of the rigged Hilbert space \mathcal{H}_{time} , being normalized to $\delta(0)$. When $H(\hat{q}, \hat{p})$ acts it "calculates" E_i from $|E_i\rangle \in \mathcal{H}_{space}$, while \hat{s} just "reads" E_i from $|E_i\rangle \in \mathcal{H}_{time}$. When $|E_i\rangle \in \mathcal{H}_{time}$ is taken in the time representation $\langle t|E_i\rangle$, then $e^{\frac{1}{i\hbar}E_it}$ appears. This term emerges in standard QM when the solution of Schroedinger equation for a system in the eigenstate of Hamiltonian is discussed. So, it could be said that the phase factor time dependence of stationary state in standard QM actually is the time representation of $|E_i\rangle \in \mathcal{H}_{time}$.

Within this approach it holds:

$$[H(\hat{q}, \hat{p}) \otimes \hat{I} + \hat{I} \otimes \hat{s}, \hat{I} \otimes \hat{t}] = i\hbar, \quad (21)$$

which means that Heisenberg uncertainty relation for total Hamiltonian and time holds. However, what one measures is not the total Hamiltonian, but its "space" part $H(\hat{q}, \hat{p}) \otimes \hat{I}$, which commutes with time operator $\hat{I} \otimes \hat{t}$. So, one may wonder is it possible to measure energy and time and to find them with sharp values simultaneously. Assuming that it is possible, it would mean that the system under consideration is in state:

$$|E_i\rangle\langle E_i| \otimes |t_a\rangle\langle t_a|, \quad (22)$$

for some E_i and t_a . But, states of this form do not satisfy dynamical equation. Only the states given by (17) are possible states of physical system and, within their second factor, eigenstates $|s_i\rangle$ of \hat{s} appear. Since $\langle s_i|t_a\rangle \neq 1$, one can find t_a only with some probability different from one. Therefore, there would be dispersion of time for every system that evolves under the

action of some conventional Hamiltonian $H(\hat{q}, \hat{p})$ of QM. In other words, dynamical equation of QM precludes measurements with sharp values of time. Measurement of time on a system in physically meaningful states, which are those satisfying von Neumann equation, can not deduce exact time. Measurement of energy, resulting or not in sharp value, is unrelated to this. Of course, the mean value of $H(\hat{q}, \hat{p}) \otimes \hat{I}$ for a system in the state (17), calculated according to:

$$\frac{\text{Tr}((H(\hat{q}, \hat{p}) \otimes \hat{I}) \hat{\rho})}{\text{Tr} \hat{\rho}},$$

is same as the mean value of this Hamiltonian calculated in standard QM (where \mathcal{H}_{time} is not taken into account).

4 Concluding Remarks

The variable s was not introduced artificially in CM. Namely, t has to have conjugate variable, which means that these two should have non vanishing Poisson bracket. Then, there should be $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ within some new Poisson bracket since the ordinary Poisson bracket contains derivatives $\frac{\partial}{\partial q}$ and $\frac{\partial}{\partial p}$ that annihilate both t and s . The Liouville equation, where "space" Poisson bracket and $\frac{\partial}{\partial t}$ are already present, lead one not to introduce, but to uncover the exact form of "time" Poisson bracket.

The intention here was to respect (more strictly than usually) the request to have time and coordinate on an equal footing. The generalized Poisson bracket (6), reexpressed Liouville equation (7) and its operator version (14) are the results of this intention. On the other hand, (14) offers the possibility to unify classical and quantum mechanics. Namely, within some generalized framework one can introduce h -dependent ($0 \leq h \leq h_o$) operators \hat{q}_h , \hat{p}_h , \hat{t}_h and \hat{s}_h , all of which become commutative for $h = 0$, representing then CM variables, and noncommutative for $h = h_o$, resembling QM. Within that framework, both CM and QM will have the same algebraic structure - the above mentioned symmetrized product, Lie bracket - operator version of the above given generalized Poisson bracket, and dynamical equation - the operator version of generalized Liouville equation. Then, classical and quantum mechanics will become completely equal regarding their structures and all the difference will rest on the commutativity of involved operators.

Moreover, that framework will offer the possibility to connect Newton and Schroedinger equations since they will follow from one and the same - the operator version of generalized Liouville equation.

The existence of the self adjoint operator \hat{s} with continuous spectrum and its presence in total Hamiltonian is followed by time-Hamiltonian uncertainty relation. On the other hand, uncertainty of time is the consequence of dynamics. As the solutions of Schroedinger equation in standard QM, the solutions of dynamical equation here time have only within the phase factors, the consequence of which is its uncertainty. (Present approach only offers more transparent argumentation on the impossibility to comprehend exact time.) Of course, this similarity, as well as the others, does not come as surprise since one of the major requests of this approach was to keep all important features of standard QM formalism.

Now, one can approach to the operator of time in reversed order. Namely, on the LHS of Schroedinger equation (20) one can recognize $i\hbar\frac{\partial}{\partial t}$ as the time representation of the operator that is conjugate to time. Then, there should exist the operator of time, which is t within the same representation. Finally, one should take \mathcal{H}_{time} , beside \mathcal{H}_{space} , where these two operators act since the time and coordinate within $\psi(q, t)$ are mutually independent. Taken in this way, it appears that the operator of time was implicitly present in QM all the time, but it has been just unnoticed.

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